

# THE CLASSIFICATION OF QUASI-ALTERNATING MONTESINOS LINKS

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**ABSTRACT.** In this note, we complete the classification of quasi-alternating Montesinos links. We show that the quasi-alternating Montesinos links are precisely those identified independently by Qazaqzeh-Chbili-Qublan and Champanerkar-Ording. A consequence of our proof is that a Montesinos link  $L$  is quasi-alternating if and only if its double branched cover is an L-space, and bounds both a positive definite and a negative definite 4-manifold with vanishing first homology.

## 1. INTRODUCTION

Quasi-alternating links were defined by Ozsváth-Szabó [OS05, Definition 3.1] as a natural generalisation of the class of alternating links.

**Definition 1.** *The set  $\mathcal{Q}$  of quasi-alternating links is the smallest set of links satisfying the following:*

- *The unknot  $U$  belongs to  $\mathcal{Q}$ .*
- *If  $L$  is a link with a diagram containing a crossing  $c$  such that*
  - (1) *both smoothings  $L_0$  and  $L_1$  of the link  $L$  at the crossing  $c$ , as in Figure 1, belong to  $\mathcal{Q}$ ,*
  - (2)  *$\det(L_0), \det(L_1) \geq 1$ , and*
  - (3)  *$\det(L) = \det(L_0) + \det(L_1)$ ,**then  $L$  is in  $\mathcal{Q}$ . The crossing  $c$  is called a quasi-alternating crossing.*

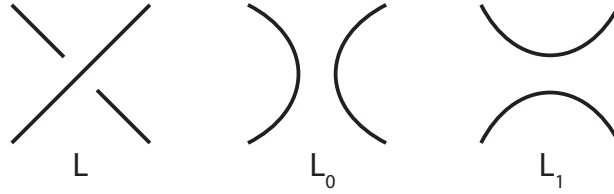


FIGURE 1.  $L$  and its two resolutions  $L_0$  and  $L_1$  in a neighbourhood of  $c$ .

Ozsváth-Szabó showed that the class of non-split alternating links is contained in  $\mathcal{Q}$  [OS05, Lemma 3.2]. Moreover, quasi-alternating links share a number of properties with alternating links, we list a few of these. For a quasi-alternating link  $L$ :

- (i)  $L$  is homologically thin for both Khovanov homology and knot Floer homology [MO08].
- (ii) The double branched cover  $\Sigma(L)$  of  $L$  is an L-space [OS05, Proposition 3.3].
- (iii) The 3-manifold  $\Sigma(L)$  bounds a smooth negative definite 4-manifold  $W$  with  $H_1(W) = 0$  [OS05, Proof of Lemma 3.6].

For some further properties see [LO15], [QC15], [Ter15] and [ORS13, Remark after Proposition 5.2].

Due to their recursive definition, it is difficult in general to determine whether or not a link is quasi-alternating. For example, there still remain examples of 12-crossing knots with unknown quasi-alternating status [Jab14]. Champanerker-Kofman [CK09] showed that the quasi-alternating property is preserved by replacing a quasi-alternating crossing with an alternating rational tangle. They used this to determine an infinite family of quasi-alternating pretzel links, which Greene later showed is the complete set of quasi-alternating pretzel links [Gre10].

Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerker-Ording [CO15] independently generalised the sufficient conditions on pretzel links to obtain an infinite family of quasi-alternating Montesinos links. This family includes all examples of quasi-alternating Montesinos links found by Widmer [Wid09]. Furthermore, it was conjectured by Qazaqzeh-Chbili-Qublan that this family is the complete set of quasi-alternating Montesinos links. We mention that Watson [Wat11] gave an iterative surgical construction for constructing all quasi-alternating Montesinos links.

Our main result is the following theorem which states that the quasi-alternating Montesinos links are precisely those found by Qazaqzeh-Chbili-Qublan [QCQ15] and Champanerker-Ording [CO15]:

**Theorem 1.** *Let  $L = M(e; t_1, \dots, t_p)$  be a Montesinos link in standard form, that is, where  $t_i = \frac{\alpha_i}{\beta_i} > 1$  and  $\alpha_i, \beta_i > 0$  are coprime for all  $i$ . Then  $L$  is quasi-alternating if and only if*

- (1)  $e < 1$ , or
- (2)  $e = 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ , or
- (3)  $e > p - 1$ , or
- (4)  $e = p - 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} < \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ .

As a corollary of our proof we obtain the following characterisation of the Montesinos links  $L$  which are quasi-alternating in terms of the double branched cover  $\Sigma(L)$ :

**Corollary 1.** *A Montesinos link  $L$  is quasi-alternating if and only if*

- (1)  $\Sigma(L)$  is an  $L$ -space, and
- (2) there exist a smooth negative definite 4-manifold  $W_1$  and a smooth positive definite 4-manifold  $W_2$  with  $\partial W_i = \Sigma(L)$  and  $H_1(W_i) = 0$  for  $i = 1, 2$ .

In light of this corollary, Theorem 1 can also be seen as a classification of the  $L$ -space Seifert fibered spaces over  $S^2$  which bound both positive and negative definite 4-manifolds with vanishing first homology. To what extent Corollary 1 generalises to non-Montesinos links remains an interesting question.

Some necessary conditions to be quasi-alternating in terms of the rational parameters of a Montesinos link were obtained in [QCQ15] and [CO15] based on the fact that a quasi-alternating link is homologically thin. Further conditions are described in [CO15] coming from the fact that the double branched cover of a quasi-alternating link is an  $L$ -space. Some additional restrictions were found in [QC15].

Our approach follows that of Greene [Gre10] on the determination of quasi-alternating pretzel links. One of Greene's main ideas is as follows. Suppose  $L$  is a quasi-alternating Montesinos link such that  $\Sigma(L)$  is the oriented boundary of the standard negative definite

plumbing  $X^4$ . Since the property of being quasi-alternating is closed under reflection, by property (iii) above,  $-\Sigma(L) = \Sigma(\bar{L})$  bounds a negative definite 4-manifold  $W$  with  $H_1(W) = 0$ . By Donaldson's theorem [Don87], the smooth closed negative definite 4-manifold  $X \cup W$  has diagonalisable intersection form. Hence,  $H_2(X)/\text{Tors} \hookrightarrow H_2(X \cup W)/\text{Tors}$  is an embedding of the intersection lattice of  $X$  into the standard negative diagonal lattice. Moreover, using that  $H_1(W)$  is torsion free, it is shown that if  $A$  is a matrix representing the lattice embedding then  $A^T$  must be surjective.

When  $L$  is a pretzel link of a certain form, Greene analyses the possible embeddings of the intersection lattice of  $X$  into a negative diagonal lattice and shows that the aforementioned surjectivity condition cannot hold, and hence the link cannot be quasi-alternating. Our main contribution is to argue for more general Montesinos links  $L$  that there is no lattice embedding for which  $A^T$  is surjective. Key to our argument are some results on lattice embeddings by Lecuona-Lisca [LL11]. The condition we obtain combined with an obstruction based on  $\Sigma(L)$  being an L-space leads to the precise necessary conditions to complete the determination of quasi-alternating Montesinos links.

## 2. PRELIMINARIES

We briefly recall some material on Montesinos links and plumbings. See [CO15] or [BZH14] for further detail on Montesinos links, and [NR78] for more on plumbings. The Montesinos link  $M(e; t_1, \dots, t_p)$ , where  $t_i = \frac{\alpha_i}{\beta_i} \in \mathbb{Q}$  with  $\alpha_i > 1$  and  $\beta_i$  coprime integers, and  $e$  is an integer, is given by the diagram in Figure 2, where rational tangles are inserted represented by the rationals  $t_i$ . The 0 rational tangle is shown in Figure 3. Introducing an additional positive (resp. negative) half-twist to the bottom of a  $a/b$  rational tangle produces a rational tangle represented by  $a/b + 1$  (resp.  $a/b - 1$ ), see Figure 3. Rotating (in either direction) a rational tangle represented by  $t \in \mathbb{Q} \cup \{1/0\}$  by 90 degrees produces the rational tangle represented by  $-1/t$ . The rational tangle represented by any  $a/b \in \mathbb{Q} \cup \{1/0\}$  can be obtained from the 0 rational tangle by a sequence of these two operations. See [Cro04] for a more thorough treatment of rational links.

We note that with our conventions for a Montesinos link  $M(e; t_1, \dots, t_p)$ , the integer  $e$  has opposite sign to that used by Champanerkar-Ording [CO15], and agrees with that of Qazaqzeh-Chbili-Qublan [QCQ15] and Greene [Gre10].

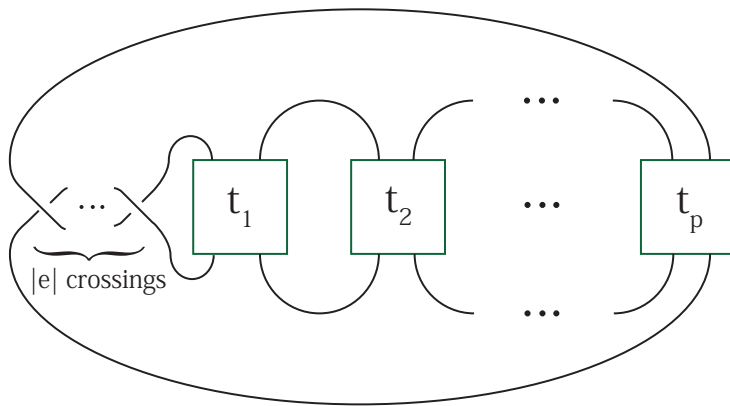


FIGURE 2. The Montesinos link  $M(e; t_1, \dots, t_p)$  where a box labelled  $t_i$  represents a rational tangle corresponding to  $t_i$ . The crossing type of the  $|e|$  crossings depends on the sign of  $e$ , with  $e > 0$  shown.

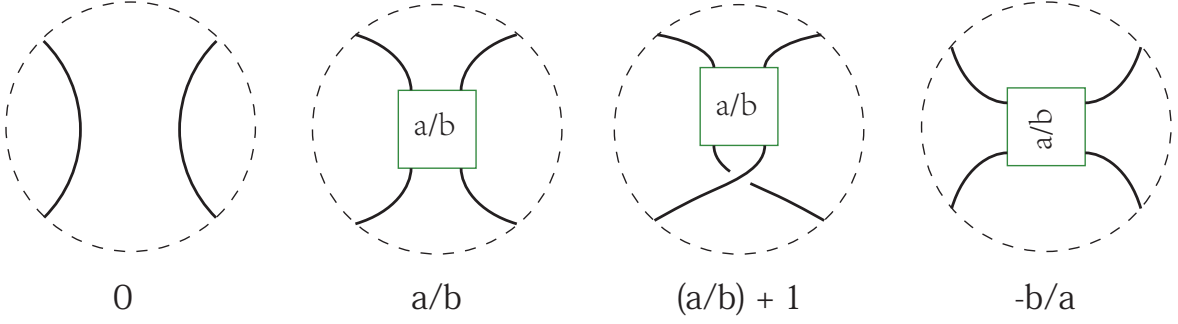


FIGURE 3. From left to right: the 0 rational tangle, an abstract representation of a  $a/b$  rational tangle, the  $\frac{a}{b} + 1$  rational tangle, and the  $-b/a$  rational tangle.

The Montesinos link  $M(e; t_1, \dots, t_p)$  is isotopic to  $M(e + 1; t_1, \dots, t_{i-1}, t'_i, t_{i+1}, \dots, t_p)$  where  $t'_i = \frac{\alpha_i}{\beta_i + \alpha_i}$ , and is also isotopic to  $M(e - 1; t_1, \dots, t_{i-1}, t'_i, t_{i+1}, \dots, t_p)$ , where  $t'_i = \frac{\alpha_i}{\beta_i - \alpha_i}$ . Hence, a Montesinos link is isotopic to one in *standard form*, that is, of the form  $M(e; t_1, \dots, t_p)$  where  $t_i > 1$  for all  $i$ .

Let  $L = M(e; t_1, \dots, t_p)$  where  $t_i < -1$  for all  $i$ . Note that any Montesinos link can be put into this form. For each  $i$ , there is a unique continued fraction expansion

$$t_i = [a_1^i, \dots, a_{h_i}^i] := a_1^i - \frac{1}{a_2^i - \frac{1}{\ddots a_{h_i-1}^i - \frac{1}{a_{h_i}^i}}},$$

where  $h_i \geq 1$  and  $a_j^i \leq -2$  for all  $j \in \{1, \dots, h_i\}$ .

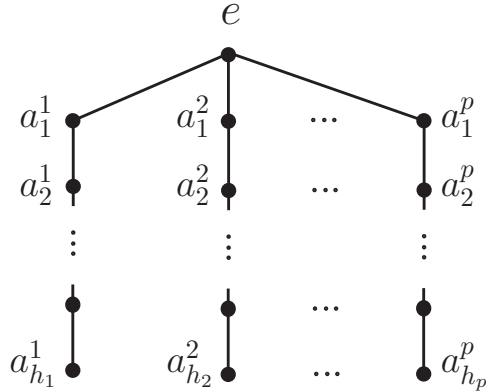


FIGURE 4. The weighted star-shaped plumbing graph  $\Gamma$ .

The double branched cover  $\Sigma(L)$  of  $L$  is the oriented boundary of the 4-dimensional plumbing  $X_\Gamma$  of  $D^2$ -bundles over  $S^2$  described by the weighted star-shaped graph  $\Gamma$  shown in Figure 4. We call  $\Gamma$  the standard star-shaped plumbing graph for  $L$ . The  $i$ th leg of  $\Gamma$  corresponding to  $t_i$  is the linear subgraph generated by the vertices labelled with weights  $a_1^i, \dots, a_{h_i}^i$ . The degree  $p$  vertex labelled with weight  $e$  is called the central vertex. Denote the vertices of  $\Gamma$  by  $v_1, v_2, \dots, v_k$ . The zero-sections of the  $D^2$ -bundles over  $S^2$  corresponding to each of  $v_1, \dots, v_k$  in the plumbing together form a natural spherical basis for  $H_2(X_\Gamma)$ . With respect to this basis, the intersection form of  $X_\Gamma$  is given by the matrix  $Q_\Gamma$  with entries  $Q_{ij}$ ,  $1 \leq i, j \leq k$  given by

$$Q_{ij} = \begin{cases} w(v_i), & \text{if } i = j \\ 1, & \text{if } v_i \text{ and } v_j \text{ are connected by an edge,} \\ 0, & \text{otherwise} \end{cases}$$

where  $w(v_i)$  is the weight of vertex  $v_i$ . We call  $(\mathbb{Z}^k, Q_\Gamma)$  the intersection lattice of  $X_\Gamma$  (or of  $\Gamma$ ).

### 3. RESULTS

The following theorem giving sufficient conditions for a Montesinos link to be quasi-alternating is Theorem 5.3 of [CO15], and also Theorem 3.5 of [QCQ15].

**Theorem 2** ([CO15], [QCQ15]). *Let  $L = M(e; t_1, \dots, t_p)$  be a Montesinos link in standard form, that is, where  $t_i = \frac{\alpha_i}{\beta_i} > 1$  and  $\alpha_i, \beta_i > 0$  are coprime for all  $i$ . Then  $L$  is quasi-alternating if*

- (1)  $e < 1$ , or
- (2)  $e = 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ , or
- (3)  $e > p - 1$ , or
- (4)  $e = p - 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} < \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ .

The goal of this section is to prove Theorem 1 which states that the sufficient conditions given in Theorem 2 for a Montesinos link to be quasi-alternating are also necessary conditions.

**Lemma 1.** *Let  $L = M(e; t_1, \dots, t_p)$ ,  $p \geq 3$ , be a Montesinos link in standard form, i.e. where  $t_i = \frac{\alpha_i}{\beta_i} > 1$  and  $\alpha_i, \beta_i > 0$  are coprime for all  $i$ . Suppose that  $e \leq p - 2$  and  $e - \sum_{i=1}^p \frac{1}{t_i} > 0$  (in particular  $e \geq 1$ ). Then  $\Sigma(L)$  is not an L-space, and therefore  $L$  is not quasi-alternating.*

*Proof.* The reflection of  $L$  is given by  $\bar{L} = M(e'; t'_1, \dots, t'_p) = M(-e; -t_1, \dots, -t_p)$ . The space  $\Sigma(\bar{L})$  is the oriented boundary of a plumbing  $X_\Gamma$  corresponding to the standard star-shaped plumbing graph  $\Gamma$  for  $\bar{L}$ . Since  $e' - \sum_{i=1}^p \frac{1}{t'_i} = -\left(e - \sum_{i=1}^p \frac{1}{t_i}\right) < 0$ , by [NR78, Theorem 5.2],  $X_\Gamma$  has negative definite intersection form.

Since  $X_\Gamma$  is negative definite and  $\Gamma$  is almost-rational, by [Ném05, Theorem 6.3] we have that  $\Sigma(\bar{L})$  is an L-space if and only if  $X_\Gamma$  is a rational surface singularity (more generally, see [Ném15]). Note that  $\Gamma$  is almost-rational since by sufficiently decreasing the weight of the central vertex we obtain a plumbing graph satisfying  $-w(v) \geq \deg(v)$  for all vertices  $v$ , where  $w(v)$  denotes the weight of  $v$ , and such a graph is rational (for details see [Ném05, Example 8.2(3)]).

Laufer's algorithm [Lau72, Section 4] can be used to determine whether the negative definite plumbing  $X_\Gamma$  is a rational surface singularity as follows. Let  $v_1, \dots, v_k$  be the vertices of  $\Gamma$  and for  $i \in \{1, \dots, k\}$ , let  $[\Sigma_{v_i}] \in H_2(X_\Gamma)$  be the spherical class naturally associated to  $v_i$ . The algorithm is as follows (see [Sti08, Section 3] for a similar formulation).

- (1) Let  $K_0 = \sum_{i=1}^k [\Sigma_{v_i}] \in H_2(X_\Gamma)$ .

- (2) In the  $i$ th step, consider the pairings  $K_i \cdot v_j = Q(K_i, v_j)$ , for  $j \in \{1, \dots, k\}$ . If for some  $j$  the pairing is at least 2 then the algorithm stops and  $X_\Gamma$  is not a rational surface singularity. If for some  $j$ , the pairing is equal to 1, then arbitrarily pick such a  $j$ , set  $K_{i+1} = K_i + [\Sigma_{v_j}]$  and go to the next step. Otherwise all pairings are non-positive, the algorithm stops and  $X_\Gamma$  is a rational surface singularity.

Applying Laufer's algorithm to  $X_\Gamma$ , we claim that the algorithm terminates at the 0<sup>th</sup> step. To see this, note that for  $v$  the central vertex of  $\Gamma$ ,  $Q(K_0, v) = p - e$  (each vertex adjacent to  $v$  contributes 1, the central vertex contributes  $-e$ ). By assumption  $e \leq p - 2$  so  $Q(K_0, v) = p - e \geq 2$ . Hence, the algorithm terminates, we conclude that  $X_\Gamma$  is not a rational surface singularity and hence  $\Sigma(\bar{L})$  is not an L-space. Therefore  $\Sigma(L)$  is not an L-space.  $\square$

**Lemma 2** ([Gre10, Lemma 2.1]). *Suppose that  $X$  and  $W$  are a pair of 4-manifolds,  $\partial X = -\partial W = Y$  is a rational homology sphere, and  $H_1(W)$  is torsion-free. Express the map  $H_2(X)/\text{Tors} \rightarrow H_2(X \cup W)/\text{Tors}$  with respect to a pair of bases by the matrix  $A$ . This map is an inclusion, and  $A^T$  is surjective. In particular, if some  $k$  rows of  $A$  contain all the non-zero entries of some  $k$  of its columns, then the induced  $k \times k$  minor has determinant  $\pm 1$ .*

**Lemma 3** ([LL11, Lemma 3.1]). *Suppose  $-1/r = [a_1, \dots, a_n]$  and  $-1/s = [b_1, \dots, b_m]$  where  $r + s = 1$ . Consider a weighted linear graph  $\Psi$  having two connected components,  $\Psi_1$  and  $\Psi_2$ , where  $\Psi_1$  consists of  $n$  vertices  $v_1, \dots, v_n$  with weights  $a_1, \dots, a_n$  and  $\Psi_2$  of  $m$  vertices  $w_1, \dots, w_m$  with weights  $b_1, \dots, b_m$ . Moreover, suppose that there is an embedding of the lattice  $(\mathbb{Z}^{n+m}, Q_\Psi)$  into  $(\mathbb{Z}^k, -\text{Id})$ , with basis  $e_1, \dots, e_k$ . For  $S$  a subset of vertices of  $\Psi$ , define*

$$U_S = \{e_i \mid e_i \cdot v \neq 0 \text{ for some } v \in S\}.$$

*Suppose further that  $e_1 \in U_{v_1} \cap U_{w_1}$  and  $U_\Psi = \{e_1, \dots, e_k\}$ . Then  $U_{\Psi_1} = U_{\Psi_2}$  and  $k = n + m$ .*

**Lemma 4** ([LL11, Lemma 3.2]). *Let  $-1/r = [a_1, \dots, a_n]$  and  $-1/s = [b_1, \dots, b_m]$  be such that  $r + s \geq 1$ . Then there exists  $n_0 \leq n$  and  $m_0 \leq m$  such that  $-1/r_0 = [a_1, \dots, a_{n_0}]$  and  $-1/s_0 = [b_1, \dots, b_{m_0}]$  satisfy  $r_0 + s_0 = 1$ .*

**Theorem 1.** *Let  $L = M(e; t_1, \dots, t_p)$  be a Montesinos link in standard form, that is, where  $t_i = \frac{\alpha_i}{\beta_i} > 1$  and  $\alpha_i, \beta_i > 0$  are coprime for all  $i$ . Then  $L$  is quasi-alternating if and only if*

- (1)  $e < 1$ , or
- (2)  $e = 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} > \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ , or
- (3)  $e > p - 1$ , or
- (4)  $e = p - 1$  and  $\frac{\alpha_i}{\alpha_i - \beta_i} < \frac{\alpha_j}{\beta_j}$  for some  $i, j$  with  $i \neq j$ .

*Proof.* If one of the conditions (1)-(4) is satisfied then  $L$  is quasi-alternating by Theorem 2, thus it suffices to show that if none of the conditions are satisfied then  $L$  is not quasi-alternating. Thus, assume none of the conditions are satisfied, in particular  $p \geq 2$ .

By [CO15, Proposition 4.1] we have that

$$\det(L) = \left| \alpha_1 \dots \alpha_p \left( e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i} \right) \right|.$$

If  $p = 2$ , since none of the conditions are satisfied we must have  $e = 1$  and  $\frac{\alpha_1}{\alpha_1 - \beta_1} = \frac{\alpha_2}{\beta_2}$ . Hence,  $\det(L) = |\alpha_1 \alpha_2 (1 - \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2})| = 0$ , and so  $L$  is not quasi-alternating (in fact  $L$



must be the two component unlink). For the remainder of the argument we assume that  $p \geq 3$ , and  $\det(L) \neq 0$ , that is,  $e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i} \neq 0$ .

First consider the case  $1 < e < p - 1$ . The reflection of  $L$  is given by

$$\bar{L} = M \left( -e, -\frac{\alpha_1}{\beta_1}, \dots, -\frac{\alpha_p}{\beta_p} \right) = M \left( p - e, \frac{\alpha_1}{\alpha_1 - \beta_1}, \dots, \frac{\alpha_p}{\alpha_p - \beta_p} \right),$$

where the latter is written in standard form and  $1 < p - e < p - 1$ . Moreover, we see that a reflection reverses the sign of  $e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i}$  and thus by a reflection if necessary we may assume that  $e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i} > 0$ . Then by Lemma 1,  $\Sigma(L)$  is not an L-space, so  $L$  is not quasi-alternating.

It remains to consider the cases  $e = 1$  and  $e = p - 1$ . By a reflection if necessary we may assume that  $e = 1$ . Note that conditions (2) and (4) are equivalent under a reflection. We assume that condition (2) is not satisfied. We need to prove that this implies that  $L$  is not quasi-alternating. If  $e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i} > 0$  then by Lemma 1,  $\Sigma(L)$  is not an L-space, and therefore  $L$  is not quasi-alternating.

Otherwise  $e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i} < 0$ . We have that

$$L = M \left( 1; \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_p}{\beta_p} \right) = M \left( 1 - p; \frac{\alpha_1}{\beta_1 - \alpha_1}, \dots, \frac{\alpha_p}{\beta_p - \alpha_p} \right),$$

where  $\frac{\alpha_i}{\beta_i - \alpha_i} < -1$  for all  $i$ .

Therefore the double branched cover  $\Sigma(L)$  of  $L$  is the boundary of a plumbing 4-manifold  $X_\Gamma$  on the standard star-shaped planar graph  $\Gamma$  with central vertex of weight  $-(p - 1)$  and legs corresponding to the fractions  $\frac{\alpha_i}{\beta_i - \alpha_i}$ ,  $i \in \{1, \dots, p\}$ . Our assumption that  $e - \sum_{i=1}^p \frac{\beta_i}{\alpha_i} < 0$  implies that  $X_\Gamma$  is negative definite [NR78, Theorem 5.2]. Suppose for the sake of contradiction that  $L$  is quasi-alternating. Then  $\bar{L}$  is quasi-alternating and  $-\Sigma(L) = \Sigma(\bar{L})$  bounds a negative definite 4-manifold  $W$  with  $H_1(W) = 0$  [OS05, Proof of Lemma 3.6]. By Donaldson's theorem [Don87], the smooth closed negative definite 4-manifold  $X_\Gamma \cup W$  has diagonalisable intersection form. Thus, the map  $H_2(X_\Gamma)/\text{Tors} \hookrightarrow H_2(X_\Gamma \cup W)/\text{Tors}$  induced by the inclusion map is an embedding of the intersection lattice  $(\mathbb{Z}^k, Q_\Gamma)$  of  $X_\Gamma$  into the standard negative diagonal lattice  $(\mathbb{Z}^n, -\text{Id})$  for some  $n$ . Denote by  $e_1, \dots, e_n$  a basis for  $(\mathbb{Z}^n, -\text{Id})$ .

We use the lattice embedding to identify elements of  $(\mathbb{Z}^k, Q_\Gamma)$  with their image in  $(\mathbb{Z}^n, -\text{Id})$ . The central vertex  $v$  of  $\Gamma$  has weight  $-(p - 1)$ , and so  $v \cdot e_i \neq 0$  for at most  $p - 1$  values of  $i \in \{1, \dots, n\}$ . Thus, by applying an automorphism if necessary, we may assume that  $v$  pairs non-trivially with precisely  $e_1, \dots, e_m$  where  $m \leq p - 1$ . Since there are  $p$  legs, by the pigeonhole principle there must exist some  $e_j$ , where  $j \in \{1, \dots, m\}$ , and two distinct vertices  $v_1, v_2$  adjacent to  $v$  with  $v_1 \cdot e_j \neq 0$  and  $v_2 \cdot e_j \neq 0$ . Without loss of generality we assume that  $j = 1$  and that for  $i \in \{1, 2\}$ , the vertex  $v_i$  belongs to the  $i$ th leg of  $\Gamma$ , i.e. corresponding to the fraction  $\frac{\alpha_i}{\beta_i - \alpha_i}$ .

Since we are assuming condition (2) does not hold, we have that  $\frac{\alpha_i}{\alpha_i - \beta_i} \leq \frac{\alpha_j}{\beta_j}$  for all  $i, j$  with  $i \neq j$ . In particular, we have  $\frac{\alpha_1}{\alpha_1 - \beta_1} \leq \frac{\alpha_2}{\beta_2}$ . Rearranging this gives  $\frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \leq 1$ . Note that the two legs correspond to the fractions  $-1/r := -\frac{\alpha_1}{\alpha_1 - \beta_1} = [a_1^1, \dots, a_{h_1}^1]$  and  $-1/s := -\frac{\alpha_2}{\alpha_2 - \beta_2} = [a_1^2, \dots, a_{h_2}^2]$ ,  $r, s \in \mathbb{Q}$ , where our notation is as in Section 2. Thus, we have that  $r + s = 2 - \frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2} \geq 1$ . Since  $r + s \geq 1$ , by Lemma 4 there exist  $h'_1 \leq h_1$  and  $h'_2 \leq h_2$  such that  $-1/r_0 = [a_1^1, \dots, a_{h'_1}^1]$  and  $-1/s_0 = [a_1^2, \dots, a_{h'_2}^2]$  with  $r_0 + s_0 = 1$ .

Let  $\Psi$  be the union of the linear graph containing the first  $h'_1$  vertices of the first leg (where we count vertices in a leg starting away from the central vertex), and the linear graph containing the first  $h'_2$  vertices of the second leg. By restricting our embedding of  $(\mathbb{Z}^k, Q_\Gamma)$ , we have an embedding of the sublattice corresponding to  $\Psi$  into  $(\mathbb{Z}^n, -\text{Id})$ . The image of this embedding is contained in a sublattice  $(\mathbb{Z}^d, -\text{Id})$  of  $(\mathbb{Z}^n, -\text{Id})$ , where  $d$  is chosen to be minimal. Then we can apply Lemma 3 which implies that  $d = h'_1 + h'_2$ .

Let  $A$  be the matrix representing the embedding  $(\mathbb{Z}^k, Q_\Gamma)$  into  $(\mathbb{Z}^n, -\text{Id})$ . Then the  $h'_1 + h'_2$  columns of  $A$  corresponding to the vertices of  $\Psi$  are supported in  $d = h'_1 + h'_2$  rows of  $A$  corresponding to the  $d$ -dimensional sublattice of  $(\mathbb{Z}^n, -\text{Id})$ . Denote this  $d \times d$  minor by  $B$ . Then  $-B^T B$  is a matrix for the intersection form of the plumbing corresponding to  $\Psi$ . Hence  $-B^T B$  is a presentation matrix for  $H_1(Y)$  where  $Y$  is the boundary of the (disconnected) plumbing corresponding to  $\Psi$ . The 3-manifold  $Y$  is the disjoint union of two lens spaces, each given by surgery on the unknot with framings  $-1/r_0 < -1$  and  $-1/s_0 < -1$  respectively. Therefore  $|\det(B)|^2 = |H_1(Y)| > 1$  contradicting Lemma 2. Thus,  $L$  is not quasi-alternating.  $\square$

**Corollary 1.** *A Montesinos link  $L$  is quasi-alternating if and only if*

- (1)  $\Sigma(L)$  is an  $L$ -space, and
- (2) there exist a smooth negative definite 4-manifold  $W_1$  and a smooth positive definite 4-manifold  $W_2$  with  $\partial W_i = \Sigma(L)$  and  $H_1(W_i) = 0$  for  $i = 1, 2$ .

*Proof.* This is a corollary of the proof of Theorem 1. Suppose first that  $L$  is quasi-alternating. By [OS05, Proposition 3.3],  $\Sigma(L)$  is an  $L$ -space. Furthermore,  $\Sigma(L)$  must bound a negative definite 4-manifold  $W_1$  with  $H_1(W_1) = 0$  [OS05, Proof of Lemma 3.6]. Applying this to the reflection of  $L$  which is also quasi-alternating, we get that  $\Sigma(L)$  also bounds a positive definite 4-manifold  $W_2$  with  $H_1(W_2) = 0$ . For the converse, note that these two necessary conditions are the only conditions used to obstruct a Montesinos link from being quasi-alternating in the proof of Theorem 1.  $\square$

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